Risk-sensitive Ramsey growth model

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Abstract. In this note we focus attention on risk-sensitive approach to an extended version of the Ramsey growth model. In contrast to the standard Ramsey model we assume that every splitting of production between consumption and capital accumulation is influenced by some random factor governed by transition probabilities depending on the current value of the accumulated capital and possibly on some (costly) decisions. Moreover, we assume that also some additional (expensive) interventions of the decision maker are possible for changing the depreciation rate of the capital. Finding optimal policy of the extended model can be then formulated as finding optimal policy of a highly structured Markov decision process. Unfortunately usual optimization criteria for Markov decision processes cannot reflect variability-risk features of the problem. To this end, we indicate how finding policies yielding maximal risk-sensitive rewards, i.e., if the stream of undiscounted one-stage rewards/costs is evaluated by an exponential utility function, can be also performed.

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JEL classification: C61, E21, E22
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1 Introduction and summary

The heart of the seminal paper of F. Ramsey [10] on mathematical theory of saving is an economy producing output from labour and capital and the task is to decide how to divide production between consumption and capital accumulation to maximize the global utility of the consumption. Ramsey's model is purely deterministic originally considered in continuous-time setting; Ramsey suggested some variational methods for finding an optimal policy how to divide the production between consumption and capital accumulation.

In the present note we formulate the Ramsey model in the discrete-time setting similarly as in the recent literature on economic growth models (see e.g. Le Van and Dana [3], Heer and Maußer [4] or Majumdar, Mitra, and Nishimura [8]). Moreover, in contrast to the standard Ramsey's model we assume that every splitting of production between consumption and capital accumulation is influenced by some random factor; in particular, governed by transition probabilities depending on the current value of the accumulated capital and possibly on some (costly) decisions. Furthermore, we assume that also some additional (expensive) interventions of the decision maker are possible for changing the depreciation rate of the capital. Finding optimal policy of this model can be then formulated as finding optimal policy of a highly structured Markov decision process. Unfortunately, usual optimization criteria for Markov decision chains as total discounted or average rewards/costs cannot reflect variability-risk features of the problem. To this end, we focus attention on policies yielding maximal risk-sensitive rewards, i.e., if the stream of undiscounted one-stage rewards/costs is evaluated by an exponential utility function.

This note is structured as follows. In Section 2, we formulate the classical Ramsey problem in the discrete-time setting. Section 3 presents an extended version of the growth model. In the extended version we assume that the development of the economy over time is described by a Markov reward chain with possible (costly) decisions. Some facts on economic decisions and utility functions are discussed in Section 4. Basic facts on optimal decisions in the extended growth model along with discretization of the model important for numerical solution are contained in Section 5, followed by Conclusions.

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2 Classical Ramsey growth model

We consider at discrete time points $t = 0, 1, \ldots$, an economy in which at each time t there are L_t (merely identical) consumers with consumption c_t per individual. The number of consumers grow very slowly in time, i.e. $L_t = L_0(1+n)^t$ for t with $\alpha := (1+n) \approx 1$. The economy produces at time t gross output Y_t using only two inputs: capital K_t and labour $L_t = L_0(1+n)^t$. A production function $F(K_t, L_t)$ relates input to output, i.e.

$$Y_t = F(K_t, L_t) \quad \text{with} \quad K_0 > 0,\ L_0 > 0 \quad \text{given.} \tag{1}$$

We assume that $F(\cdot, \cdot)$ is a strictly increasing concave twice differentiable homogeneous function of degree one, i.e. $F(\theta K, \theta L) = \theta F(K, L)$ for any $\theta \in \mathbb{R}$.

The output must be split between consumption $C_t = c_t L_t$ and gross investment I_t, i.e.

$$C_t + I_t \leq Y_t = F(K_t, L_t). \tag{2}$$

Investment I_t is used in whole (along with the depreciated capital K_t) for the capital at the next time point $t+1$. In addition, capital is assumed to depreciate at a constant rate $\delta \in (0, 1)$, so capital related to gross investment at time $t+1$ is equal to

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{3}$$

In what follows let $k_t := K_t/L_t$ be the capital per consumer at time t, and similarly let $y_t := Y_t/L_t$ be the per capita output at time t. Recalling that the production function $F(\cdot, \cdot)$ is assumed to be homogeneous of degree one, then $f(k_t) := F(k_t, 1)$ denotes the per capita production per unit time. In virtue of (2), (3) we get

$$c_t + (1+n)k_{t+1} - (1-\delta)k_t \leq y_t = f(k_t), \tag{4}$$

and if we set for simplicity $\alpha \equiv (1+n) = 1$ then (4) can be written as

$$c_t + k_{t+1} - (1-\delta)k_t \leq y_t = f(k_t). \tag{5}$$

The aim is to find a rule how to split production between consumption and capital accumulation that maximizes for the given time horizon T the utility function

$$U(c_0, \ldots, c_T) \quad \text{of a single consumer.} \tag{6}$$

Recall that the utility function $U(c_0, \ldots, c_T)$ is real, strictly increasing and concave function in all its arguments $c_0, \ldots, c_T$ and if all $c_i \equiv 0$ then also $U(c_0, \ldots, c_T) = 0$.

As we shall see later in virtue of the recursive form of the formulas (4), (5), the problem is much easier to solve if the utility function is additive, i.e. if preferences for consumption of a single consumer (resp. for the considered number of L_t consumers) are taken for the considered time horizon T in the form

$$U(c_0, \ldots, c_T) = \sum_{t=0}^{T} u(c_t) \quad (\text{resp.}\ \ \bar{U}(c_0, \ldots, c_T) = L_0 \sum_{t=0}^{T} (\alpha)^t u(c_t).) \tag{7}$$

In the above formulation we assume that the per capita production function $f(k)$ and the consumption function $u(c)$ fulfil some standard assumptions on production and consumption functions, in particular, that:

AS 1. The function $u(c) : \mathbb{R}^+ \to \mathbb{R}^+$ is twice continuously differentiable and satisfies $u(0) = 0$. Moreover, $u(c)$ is strictly increasing and concave (i.e., its derivatives satisfy $u'(\cdot) > 0$ and $u''(\cdot) < 0$) with $u'(0) = +\infty$ (so-called Inada Condition).

AS 2. The function $f(k) : \mathbb{R}^+ \to \mathbb{R}^+$ is twice continuously differentiable and satisfies $f(0) = 0$. Moreover, $f(k)$ is strictly increasing and concave (i.e., its derivatives satisfy $f'(\cdot) > 0$ and $f''(\cdot) < 0$) with $f'(0) = M < +\infty$, $\lim_{k\to\infty} f'(k) < 1$. Hence, if $f'(\cdot) > 1$ there exists k^* such that $f'(k^*) = k^*$.

Since $u(\cdot)$ is increasing (cf. assumption AS 1) in order to maximize global utility of the consumers is possible to replace (5) by the (nonlinear) difference equation

$$k_{t+1} - (1-\delta)k_t - f(k_t) = -c_t \quad \text{with } k_0 > 0 \text{ given} \tag{8}$$

or equivalently for $\tilde{f}(k) := f(k) + (1-\delta)k$ by

$$k_{t+1} - \tilde{f}(k_t) = -c_t \quad \text{with } k_0 \text{ given}, \tag{9}$$

where c_t $(t = 0, 1, \dots)$ with $c_t \in [0, f(k_{t-1})]$ is selected by the decision maker.

Up to now the system described above is purely deterministic; hence the initial capital k_0 along with the control policy c_t fully determines development of (k_t, c_t) over time.

3 Extended version of the growth model

Unfortunately, in the real-life situations also some random shocks or imprecisions should be considered. For this reason, we shall assume that for a given value of k_t we obtain the output value y_t with some uncertainty; in particular we assume that $y_t \in [f_{\min}(k_t), f_{\max}(k_t)]$ (i.e. $f_{\min}(\cdot) \le f(\cdot) \le f_{\max}(\cdot)$; AS 2 also hold for $f_\bullet(k)$). Obviously, better results can be obtained if we replace the rough estimates of y_t generated by means of $f_{\max}(k_t)$ and $f_{\min}(k_t)$ by a more detailed information on the (random) output y_t generated by the capital k_t. We assume that $k_t \le k_{\max}$, $y_t \le y_{\max}$ for $t = 0, 1, \dots$.

To this end we shall assume that in (5) $y_t = f(k_t)$ is replaced by $y_t = x(k_t)$, where $X = \{x(\cdot)\}$ is a Markov process with state space $\mathcal{I}_1 \subset \mathbb{R}$ and transition probabilities $p(y|k)$ from state $k \in \mathcal{I}_1$ to state $y \in \mathcal{I}_2 \subset \mathbb{R}$ such that $p(y_t|k_t) \gg p(y|k_t)$ for each $y \ne y_t = f(k_t)$ (obviously, $\sum_{y\in\mathcal{I}_2} p(y|k) = 1$ for each $k \in \mathcal{I}_1$). Such an extension well corresponds to the models introduced and studied in [14] and also in [4, 8]. Moreover, we assume that the current value of the total output y_t is known to the decision maker *and then the recourse decision (intervention) at a cost $\tilde{s}(y,k)$ may be taken to reach the desired value of* $k_{t+1} = y_t - c_t + (1-\delta)k_t$ for sure. If no intervention is made then the desired value k_{t+1} will be reached with probability $p(k_{t+1}|y_t) < 1$, we assume that $p(\tilde{k}|y_t) \ll p(k_{t+1}|y_t)$ for each $\tilde{k} \ne k_{t+1}$ with $p(\tilde{k}|y_t) = 0$ for $\tilde{k} > k_{t+1}$. Up to now we have assumed that the transition probabilities cannot be influenced by the decision maker. In what follows we extend the model in such a way that $p(y|k)$ will be replaced by $p(y|k,d)$ for $d \in \mathcal{D} = \{1, 2, \dots, D\}$ and some *cost,* denoted $s(k,d)$, *will be accrued to the decision* $d \in \mathcal{D}$. Similarly, $f(k)$ should be replaced by $f(k,d)$.

So the development of the considered system over time is given by the following diagram

$$k = k_t \xrightarrow[]{\substack{s(k,d)\\ p(y|k,d)}} y = y_t \xrightarrow[]{\substack{\tilde{s}(y,\tilde{k})\\ \tilde{p}(\tilde{k}|y)}} \tilde{k} = k_{t+1}$$

The above model can be also treated as a structured controlled Markov reward process X with compact state space $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ (with $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$), finite set $\mathcal{D} = \{0, 1, \dots, D\}$ of possible decisions (actions) in each state $k \in \mathcal{I}_1$, possible interventions in state $y \in \mathcal{I}_2$, and the following transition and cost structure:

- $p(y|k,d)$: transition probability $k \in \mathcal{I}_1 \to y \in \mathcal{I}_2$ if decision $d \in \mathcal{D}$ is selected,
- $s(k,d)$: cost of decision $d \in \mathcal{D}$ in state $k \in \mathcal{I}_1$,
- $r(y,\tilde{k}|k)$: one-stage reward obtained if from state $k \in \mathcal{I}_1$ state $y \in \mathcal{I}_2$ is reached and finally with respect to the current value of $y \in \mathcal{I}_2$ capital $k_{t+1} := \tilde{k}$ is available, obviously $r(y,\tilde{k}|k) = u(y + (1-\delta)k - \tilde{k})$,
- $\tilde{s}(y,\tilde{k})$: cost for intervention in state $y \in \mathcal{I}_2$ then the desired capital $\tilde{k} \in \mathcal{I}_1$ is available,
- $p(\tilde{k}|y)$: transition probability $y \in \mathcal{I}_2 \to \tilde{k} \in \mathcal{I}_1$ if no intervention is taken in state $y \in \mathcal{I}_2$; if an intervention is taken then $p(\tilde{k}|y) = 1$ for $\tilde{k} = y - c + (1-\delta)k$,
- $\bar{r}(\tilde{k}|k,d)$: expected value of the one-stage reward obtained in state k if decision $d \in \mathcal{D}$ is selected in state k and $k_{t+1} := \tilde{k}$; in particular
  $$\bar{r}(\tilde{k}|k,d) = \int_{y\in\mathcal{I}_2} p(\mathrm{d}y|k,d)[u(y + (1-\delta)k - \tilde{k})]$$
- $\tilde{r}(\tilde{k}|k,d)$: total expected reward earned by transition from state k to state $\tilde{k}$, including possible intervention in state $y \in \mathcal{I}_2$ and cost of decision $d \in \mathcal{D}$ in state $y \in \mathcal{I}_2$, i.e., $\tilde{r}(\tilde{k}|k,d) = \bar{r}(\tilde{k}|k,d) - s(k,d) - \displaystyle\int_{y\in\mathcal{I}_2} p(\,\mathrm{d}y|k,d)\bar{s}(y,\tilde{k})$.

Policy controlling development of the economy over time modelled by the Markov process X described above, say π, is a rule how to select decision (actions) in each state. Policy π is then fully identified by a sequence $\{d_\tau, \tau = 0, 1, \ldots\}$ of decisions taken in state $k \in \mathcal{I}_1$ and possible interventions in state $y \in \mathcal{I}_2$. If we restrict on stationary policies, i.e. the rules selecting actions only with respect to the current state of Markov process X, then the development of the economy over time is described by a homogeneous Markov process.

4 Economic decisions and utility functions

Economic decisions are usually based on the outcome, say ξ, as viewed by the decision maker represented by an appropriate utility function. Recall that utility functions, say $u(\cdot)$ assigning a real number to each possible outcome, are monotonically increasing and concave (cf. AS 1), i.e., we assume that larger values of outcome are preferred.

In case of stochastic systems outcome ξ is a random variable and we consider expectation of utilities assigned to (random) outcomes, i.e. the value $\mathsf{E}u(\xi)$. Certain (or certainty) equivalent, say $Z(\xi)$, is then defined by $u(Z(\xi)) := \mathsf{E}u(\xi)$ (in words, certainty equivalent is the value, whose utility is the same as the expected utility of possible outcomes).

For handling real life models decision analysts must be able to express $u(x)$ in a concrete form. Typical utility functions are:

- Linear function: $u(x) = a + bx$ where $b > 0$
- Quadratic function: $u(x) = a + bx - cx^2$ where $b > 0,\ c > 0$ for $x \in [0, b/(2c)]$.
- Logarithmic function: $u(x) = a + b \ln(x + c)$ where $b > 0,\ c \geq 0$.
- Fractional function: $u(x) = a - \dfrac{1}{x+b}$ where $b > 0,\ c > 0$
- The function: $$u(x) = \begin{cases} x^{1-a} & \text{for} \quad 0 < a < 1 \\ \ln x & \text{for} \quad a = 1 \\ -x^{1-a} & \text{for} \quad a > 1 \end{cases}$$
- Exponential function: $u(x) = -\mathrm{e}^{-ax}$ with $a > 0$

Observe that in the above family of utility functions only linear utility function (with $a = 0$) and exponential utility functions are separable and hence suitable for sequential decisions. In particular, it holds for any $x, y \in \mathbb{R}$

$$\begin{aligned} u(x+y) &= u(x) + u(y) && \text{for linear utility function with } a = 0 \\ u(x+y) &= u(x) \cdot u(y) && \text{for exponential utility function} \end{aligned}$$

Furthermore, exponential utility functions

- are the most widely used non-linear utility functions, cf. Corner, J. L. and Corner, P. D. [2],
- in most cases an appropriately chosen exponential utility function is a very good approximation for general utility function, cf. Kirkwood [6].

Unfortunately, considering stochastic models, in contrast to exponential utility functions, linear utility functions cannot reflect variability-risk features of the problem. Introducing the so-called risk aversion coefficient $\gamma \in \mathbb{R}$ then the utility $u(\xi)$ assigned to a random outcome ξ for exponential utility functions, as well as linear utility functions, can be also written in the following more compact form

$$u(\xi) = \begin{cases} (\operatorname{sign} \gamma) \exp(\gamma \xi), & \text{if } \gamma \neq 0 \\ \xi & \text{for } \gamma = 0. \end{cases} \tag{10}$$

Obviously $u^\gamma(\cdot)$ is continuous and strictly increasing, and

- convex for $\gamma > 0$, so-called risk seeking case
- concave for $\gamma < 0$, so-called risk aversion case

If exponential utility (10) is considered, then for the corresponding certainty equivalent $Z^\gamma(\xi)$ given by

$$u^\gamma(Z^\gamma(\xi)) = \mathsf{E}[(\text{sign } \gamma) \exp(\gamma\xi)]$$

we have

$$Z^\gamma(\xi) = \begin{cases} \frac{1}{\gamma} \ln\{\mathsf{E}\,[\exp(\gamma\xi)]\}, & \text{if } \gamma \neq 0 \\ \mathsf{E}[\xi] & \text{for } \gamma = 0. \end{cases} \tag{11}$$

Since exponential utility functions preserve the nice "separability" property of the linear utility functions and according to the selected "risk aversion coefficient" prefer "large" or "small" outcomes, they are extremely suitable for employing in multistage optimization problems.

5 Optimization of the extended growth model

5.1 Risk-sensitive optimality in the growth model

Suppose that in (11) $\xi^n = \sum_{i=0}^{n-1} \xi_i$ where ξ_i's is a sequence of random rewards/costs generated by the Markov reward process with parameters given in Section 3, $Z^\gamma(\xi^n)$ is the corresponding certainty equivalent and $g(\xi^n) := n^{-1} Z^\gamma(\xi^n)$ is the mean value of the certainty equivalent $Z^\gamma(\xi^n)$. Obviously, in virtue of AS 1, AS 2 also in the "stochaticized" model in Section 3 the values $r(y,k|\tilde{k})$ must be bounded, say $0 < r(y,k|\tilde{k}) < M$; similarly we assume that $0 < \tilde{s}(k,d) < m$. Hence also $\limsup_{n\to\infty} n^{-1}\tilde{r}(\tilde{k}|k,d) < M$.

For the risk neutral case (i.e. if the risk aversion coefficient $\gamma = 0$) we are interested in policies maximizing mean reward if the time horizon tends to infinity. If the state space is discrete, there is a overwhelming literature on this classical problem of stochastic dynamic programming (see e.g. [9]); attention has been also focused on models with compact state space (see e.g. [5]). In particular, since the action set is finite and one-stage rewards and costs are bounded, under some mild ergodicity conditions[1] asymptotic mean reward is independent of the starting state and maximal asymptotic mean reward, say $g^* = \lim_{n\to\infty} n^{-1}\mathsf{E}(\xi^n)$ can be found as a solution, say $(g^*, h(k^*))$, of the optimality equation (see e.g. [5])

$$g + h(k) = \max_{d\in\mathcal{D}} \int_{\tilde{k}\in\mathcal{I}_1} \bar{p}(\mathrm{d}\tilde{k}|k,d)[\tilde{r}(\tilde{k}|k,d) + h(\tilde{k})], \quad \text{for any } k, \tilde{k} \in \mathcal{I}_1 \tag{12}$$

where $\bar{p}(\tilde{k}|k,d) = \int_{y\in\mathcal{I}_2} p(\mathrm{d}y|k,d)\tilde{p}(\tilde{k}|y)$.

Unfortunately considerably less attention has been paid to so-called risk-sensitive case (where the risk aversion coefficient $\gamma \neq 0$) originally studied in the seminal paper [7] for the models with finite state and action spaces and irreducible transition probability matrices. As it was shown in the recent paper [1] under mild assumptions it can be shown that even for models with compact state space policy maximizing mean certain equivalent also exists in the family of stationary policies being a solution of the following Poisson equation

$$e^{g+h(k)} = \max_{d\in\mathcal{D}} \int_{\tilde{k}\in\mathcal{I}_1} \bar{p}(\mathrm{d}\tilde{k}|k,d) e^{[\tilde{r}(\tilde{k}|k,d)+h(\tilde{k})]}, \quad \text{for any } k, \tilde{k} \in \mathcal{I}_1. \tag{13}$$

Unfortunately, for concrete numerical calculation is seems that the best way is to discretize the model and solve the problem as risk-sensitive Markov decision process with finite state and action spaces.

5.2 Discretized growth model

In what follows, we shall assume that the values of k, and y take on only discrete values (cf. [11]). In particular, we assume that for sufficiently small $\Delta > 0$ there exists nonnegative integers $\bar{c}_t$, $\bar{k}_t$, and $\bar{y}_t$ such that for every $t = 0, 1, \ldots$ it holds:
$\bar{c}_t\Delta = c_t$, $\bar{k}_t\Delta = k_t$, and $\bar{y}_t\Delta = y_t$ with $\bar{k}_t \le K := k_{\max}/\Delta$ and similarly $\bar{y}_t \le Y := y_{\max}/\Delta$.
In what follows we assume that at time t only values $\bar{c}_t + \ell\Delta$, $\bar{k}_t + \ell\Delta$ and $\bar{y}_t + \ell\Delta$ can occur where $\ell = \{-K, -K+1, \ldots, 0, \ldots, K-1, K\}$ and integer K is given. Then the probabilities $p(y|k,d)$ and

[1] We may assume existence of state k^* specified in AS 2 such that $p(k^*|k,d) > \varepsilon > 0$ for any pair k, d.

$p(\tilde{k}|y)$ of Section 3 describing the stochastic behaviour of the extended model take on only a finite number of values.

Equations (12) and (13) then take on the form

$$g + h(k) = \max_{d\in\mathcal{D}} \sum_{\tilde{k}\in\mathcal{I}_1} \bar{p}(\tilde{k}|k,d)[\tilde{r}(\tilde{k}|k,d) + h(\tilde{k})], \qquad \text{for any } k, \tilde{k} \in \mathcal{I}_1 \tag{14}$$

$$e^{g+h(k)} = \max_{d\in\mathcal{D}} \sum_{\tilde{k}\in\mathcal{I}_1} \bar{p}(\tilde{k}|k,d) e^{[\tilde{r}(\tilde{k}|k,d)+h(\tilde{k})]}, \qquad \text{for any } k, \tilde{k} \in \mathcal{I}_1 \tag{15}$$

where $\bar{p}(\tilde{k}|k,d) = \sum_{y\in\mathcal{I}_2} p(y|k,d)\tilde{p}(\tilde{k}|y)$.

6 Conclusions

In this paper we focus are attention on extended version of the Ramsey model with exponential performance functions. It was shown that along with standard performance function (i.e. the risk neutral case with risk aversion coefficient $\gamma = 0$) also methods for risk-sensitive case (i.e. when risk aversion coefficient $\gamma \neq 0$) can be successfully employed.

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References

[1] Cavazos-Cadena, R. and Salem Silva, F.: The discounted method and equivalence of average criteria for risk-sensitive Markov decision processes on Borel space. *Appl. Mathematics Optim.* **61** (2010), 167–190.

[2] Corner, J. L. and Corner, P. D.: Characterization of decisions in decision analysis practice. J. *Oper. Res. Soc.* **46** (1995), 304–314.

[3] Dana, R.-A. and Le Van, C: *Dynamic Programming in Economics.* Kluwer, Dordrecht 2003.

[4] Heer, B. and Maußer, A.: *Dynamic General Equilibrium Modelling.* Springer, Berlin 2005.

[5] Hernández-Lerma, O. and Lasserre J.: *Discrete-Time Markov Control Processes. Basic Optimality Criteria.* Springer, New York 1995.

[6] Kirkwood, C. W.: Approximating risk aversion in decision analysis. *Decision Analysis* **1** (2004), 51–67.

[7] Howard, R. A. and Matheson, J.: Risk-sensitive Markov decision processes. *Manag. Sci.* **23** (1972), 356–369.

[8] Majamdar, M., Mitra, T., and Nishinmura, K. (eds.): *Optimization and Chaos.* Springer, Berlin 2000.

[9] Puterman, M. L.: *Markov Decision Processes – Discrete Stochastic Dynamic Programming.* Wiley, New York 1994.

[10] Ramsey, F. P.: A mathematical theory of saving. *Economic Journal* **38** (1928), 543–559.

[11] Sladký, K.: Approximations in stochastic growth models. In: *Proc. 24th Internat. Conf. Mathematical Methods in Economics 2006*, Univ. West Bohemia, Pilsen 2006, pp. 465–470.

[12] Sladký, K.: Stochastic growth models with no discounting. *Acta Oeconomica Pragensia* **15** (2007), No.4, 88–98.

[13] K. Sladký, K.: Ramsey growth model under uncertainty. In: *Proc. 27th Internat. Conf. Mathematical Methods in Economics 2009*, Faculty of Economics and Management, Czech University of Life Sciences Prague, Prague 2009, pp. 296–300.

[14] Stokey, N.-L. and Lukas, R. E. Jr.: *Recursive Methods in Economic Dynamics.* Harvard Univ. Press, Cambridge, Mass. 1989.